

Properties of Numbers

Table of Contents

- 1.0 Real numbers
- 2.0 Classification of real numbers
- 3.0 Properties of zero
- 4.0 Properties of one
- 5.0 Prime numbers
- 6.0 Signed numbers
- 7.0 Consecutive integers
- 8.0 Absolute value
- 9.0 Order of arithmetic operations
- 10.0 Concept testers

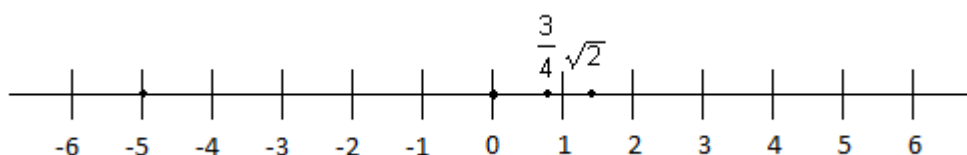
Number systems is a very important topic on the GMAT exam. The fundamental concepts and characteristics of numbers are tested and questions range from simple to complicated.

In this chapter we discuss properties and rules of numbers. We also talk about different types of numbers - including rational, irrational, and consecutive - and how to solve difficult problems using simple techniques.

1.0 Real numbers

All the numbers present on the line represented below are real numbers, including

- Integers such as 1, 0, and -20
- Fractions such as $\frac{1}{2}$, $\frac{6}{13}$, and $\frac{12}{5}$
- Rational numbers such as 5, $\frac{2}{3}$, and 5.26
- Irrational numbers such as $0.3265891\dots$, $\sqrt{2}$, and $\sqrt{3}$



Note: Real numbers do not include “imaginary” numbers such as $\sqrt{-1}$, $\sqrt{-4}$.

1.1 Properties of real numbers

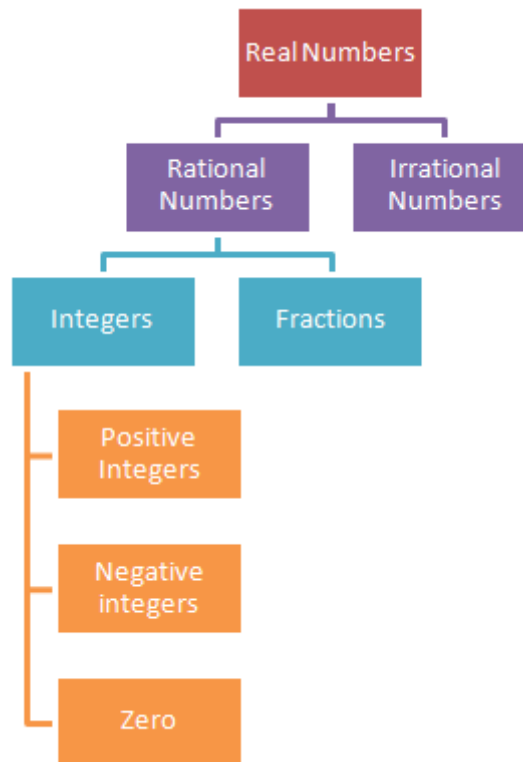
If a , b and c are real numbers, then

- $a + b$, $a - b$, ab and $\frac{a}{b}$ also real numbers ($\frac{a}{b}$ is not defined if $b = 0$)
- $a + (b + c) = (a + b) + c$
- $a(bc) = (ab)c$
- $a + b = b + a$

- $ab = ba$
- $a(b + c) = ab + ac$

2.0 Classification of real numbers

Real numbers can be classified as follows:



2.1 Rational numbers

The numbers which can be expressed in the form of a fraction $\frac{a}{b}$, where a and b are integers and b is not zero, are called rational.

Example:

$$2 = \frac{2}{1}, 0.25 = \frac{25}{100} = \frac{1}{4}, 0.36 = 0.6 = \frac{6}{10} = \frac{3}{5}$$

2.2 Irrational numbers

Irrational numbers are numbers which cannot be represented as rationals. For example, the square root of 2 is irrational. The number π (which is the ratio of any circles' circumference to its diameter) is called "pi" and is irrational; it equals *approximately* 3.141592654. The decimal version of any irrational number never terminates and never repeats a pattern, unlike rationals.

2.3 Integers

Integers are numbers having no fractional part. Integers can be positive, negative, or zero.

Examples:

$$\frac{20}{4} = 5$$

$$\frac{63}{1} = 63$$

0

Note: If the decimal place of a number contains zero then the number is considered to be an integer. The number 2.00, for example, is also an integer because its decimal part is zero.

2.4 Fractions

Fractions are numbers in the form of a/b where $b \neq 0$ provided the result of the division operation is not an integer. Thus, $12/3 = 4$ is not considered to be a fraction.

Examples:

$$\frac{12}{10} = 1.2$$

$$\frac{1}{4} = 0.25$$

$$\frac{15}{7}$$

3.0 Properties of zero

- Zero is the point on the number line having all positive numbers to its right, and all negative numbers to its left.
- Zero is neither negative nor positive.
- Zero additive identity property: addition of zero to a number results in the identical number. For instance, $n + 0 = 0 + n = n$.
- The product of a number and zero will always be zero: $0(a) = 0$.
If $ab = 0$ then either $a = 0$ or $b = 0$ or both a and b are zero.
- Division by zero is not defined; therefore, zero must not be the denominator.

4.0 Properties of “1”

- 1 is the smallest positive integer.
- 1 is a factor of all numbers.
- Multiplicative identity property: $1(a) = a$.

1 is the multiplicative identity element. The product of any number and “1” will result in the same number:

$$n \times 1 = 1 \times n = n$$

- Division of a number by 1 results in the same number, such as $\frac{a}{1} = a$.
- Is neither a prime number nor a composite number.
- Any power of 1 is 1; that is $1^n = 1$.

5.0 Prime numbers

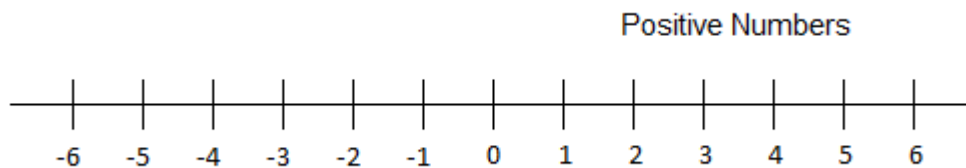
- Prime numbers are integers which are not divisible by any integer other than 1 and the number itself.
- The primes (in ascending order) are 2, 3, 5, 7, 11, 13, 17, 19, 23, and so on.
- 2 is the only even prime.

6.0 Signed numbers

Any real number is either negative, zero, or positive. Negative numbers must be preceded by their sign, $-$, but positive numbers omit their sign, $+$, as it is understood.

6.1 Positive numbers

Integers which are greater than zero are called positive integers. Positive integers are located to the right of zero on the number line, as shown below.

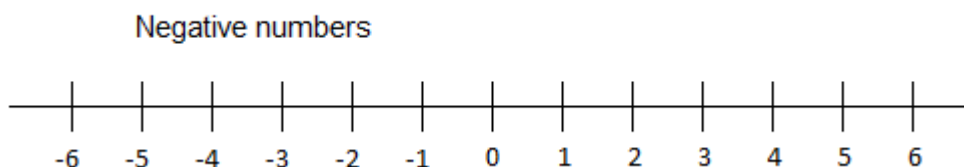


Examples:

1, 2, 30, 54

6.2 Negative numbers

Integers which are less than zero are called negative integers. Negative integers are located to the left of zero on the number line, as shown below.



Examples:

$-1, -2, -5, -10$

Note: zero is neither positive nor negative.

6.3 Properties of positive and negative numbers

The best way to describe operations (addition, subtraction, multiplication, division) on signed numbers is by illustration. In so doing, bear in mind that $-a$ is known as the *opposite of a*. Thus $-(-a) = a$; it is said that “a negative times a negative is a positive.”

- $4 + (-3) = 1$
- $4 + (-5) = -1$
- $4 + (-4) = 0$
- $4 - 3 = 1$
- $3 - 4 = -1$
- $4 - (-1) = 4 + 1 = 5$
- $-2 + (-3) = -5$ Note: $= -(2 + 3)$, that is, like signs add absolute values; then adopt the common sign.
- $4 \times (-2) = -8$
- $-4 \times (-2) = 8$
- $4/(-2) = -2$
- $-4/(-2) = 2$

7.0 Consecutive integers

The integers which are adjacent to each other on the number line are called consecutive integers.

This means that the difference between two consecutive integers is 1.

Let an integer be x , the consecutive integer after x is $(x + 1)$ and the consecutive integer after $(x + 1)$ is $((x + 1) + 1) = (x + 2)$ and so on.

The consecutive integer before x is $(x - 1)$, the consecutive integer before $(x - 1)$ is $((x - 1) - 1) = (x - 2)$ and so on.

The mathematical notation of consecutive integers is $\dots(x - 2), (x - 1), x, (x + 1), (x + 2)\dots$

Examples:

- $-2, -1, 0, 1$ are four consecutive integers beginning at negative two.
- $0, 2, 4, 6\dots$ are consecutive even integers (constant difference two).
- $1, 3, 5, 7\dots$ are consecutive odd integers (constant difference two).

The sum of n consecutive integers is divisible by n when n is odd.

Examples:

Take 5 consecutive integers 1, 2, 3, 4, and 5, whose sum is $1 + 2 + 3 + 4 + 5 = 15$.

Here, we can observe that 15 is divisible by 5

Take 3 consecutive integers 30, 31, and 32 whose sum is $30 + 31 + 32 = 93$.

93 is divisible by 3.

The sum of n consecutive integers is *not* divisible by n when n is even.

The product of n consecutive integers is divisible by $n!$ Recall that “ n factorial” or $n!$ is defined as $n \times (n - 1) \times (n - 2) \times \dots \times 3 \times 2 \times 1$ for $n > 1$, and both $0!$ and $1!$ are defined as 1.

8.0 Absolute value

Absolute value is the distance of a certain number from zero on the number line irrespective of the direction from zero. Absolute value of a number is denoted as $|n|$ and it is defined as follows:

$$|n| = n, \text{ when } n \geq 0$$

$$|n| = -n, \text{ when } n < 0. \text{ Thus, an absolute value cannot be negative.}$$

Example: $|125| = 125$, $|-96| = 96$, $|0| = 0$, $|5.236| = 5.236$, $|-1.25| = 1.25$

$$\text{Property: } |m + n| \leq |m| + |n|$$

Examples:

Case i: $m = 5$ and $n = 2$

$$|m + n| = |5 + 2| = |7| = 7$$

$$|m| + |n| = |5| + |2| = 5 + 2 = 7$$

$$|m + n| = |m| + |n|$$

Case ii: $m = 5$ and $n = -2$

$$|m + n| = |5 - 2| = |3| = 3$$

$$|m| + |n| = |5| + |-2| = 5 + 2 = 7$$

$$|m + n| < |m| + |n|$$

From case i and ii: $|m + n| \leq |m| + |n|$

9.0 Order of arithmetic operations

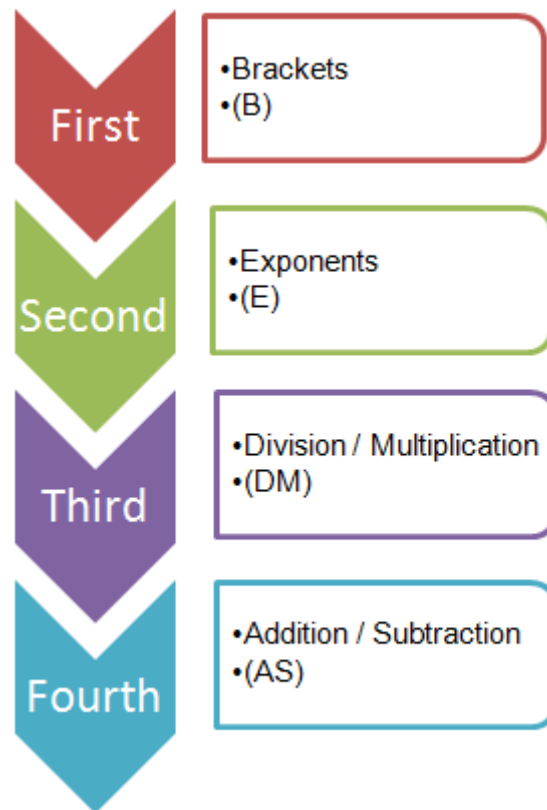
Order of arithmetic operations is a rule used to describe which operator should be used first in the given mathematical expression.

First: (P) Parenthesized expressions (includes grouped parentheses and/or square brackets)

Second: Exponents €

Third: Multiplication / Division (MD)

Fourth: Addition / Subtraction (AS)



This Order of Arithmetic operations rule is called BEDMAS—an acronym placing the choices in hierarchical order.

When an expression is evaluated, it is scanned from left to right, seeking an operator (for addition, subtraction, multiplication, division or exponentiation) and the two values that will be used in the operation.

The PEMDAS rules govern whether an operation can be performed at once. For example, in the expression $a + b \times c$, the scan encounters a and $+$ so it is known that eventually a is to be added to something. The question is, should the next value, b , be added now? PEMDAS says no, in effect, because the \times operator is higher in the hierarchy, and thus b must be multiplied by c first, and then that product is available as the second part of the deferred addition.

Example:

Evaluate $(-2)^2 \times 5 - 8$

The evaluation proceeds from left to right in a straight forward manner.

First: $(-2)^2 = -2 \times -2 = 4$

Second: $4 \times 5 = 20$

Third: $20 - 8 = 12$

Therefore: $(-2)^2 \times 5 - 8 = 12$

Example:

Evaluate $\sqrt{24 + 2 \times 3 - 1 - 2^2}$

A series of steps is shown below. First, the product 2×3 had to be calculated, and then the squaring of 2, giving $24 + 6 - 1 - 4$ as the radicand expression (under the square root symbol). That evaluation then proceeds from left to right, giving $30 - 1 - 4$ and $29 - 4$ and then 25, at which point the square root of 25 is taken giving the answer 5.

$$\sqrt{24 + 2 \times 3 - 1 - 2^2} = \sqrt{24 + 6 - 1 - 4} = \sqrt{25} = 5$$

10.0 Concept testers

1. If n is odd, what is $n(n + 1)$?

Answer:

Because n is odd, its immediate successor $(n + 1)$ is even.

Therefore, $n(n+1)$ is even.

2. If xy is even, what is $x + y$?

Answer:

Given that xy is even, then x can be even or y can be even or both x and y can be even. $x + y$ is even when both x and y are even, otherwise it is odd. But we do not know whether both x and y are even; so, we cannot determine the answer.

3. Evaluate $\frac{105}{5} + \frac{3 \times 21}{7} - 10$

Answer:

$$\frac{105}{5} + \frac{3 \times 21}{7} - 10 = -21 + \frac{63}{7} - 10 = -21 + 9 - 10 = -12 - 10 = -22.$$

It is worth noting that within the original expression contained in the square brackets, where both multiplication and division appear, the multiplication by 3 could have been deferred, in favor of dividing 21 by 7; that quotient 3 could then have been multiplied by 3, giving 9 and the overall result would have been the same, -22 . In this case, we would arrive at the answer by doing the following:

First we perform the division

$$= -21 + (3 \times 3) - 10$$

Then perform multiplication

$$= -21 + 9 - 10$$

Now perform addition

$$= -12 - 10$$

Finally perform subtraction

$$= -22$$

4. Evaluate $30 \div 15 + 3 \times 7 - 20$

Answer:

$$\frac{30}{15} + 3 \times 7 - 20 = 2 + 21 - 20 = 23 - 20 = 3.$$

5. If the average of 7 consecutive integers is 14, then find those numbers.

Answer:

Let the 7 consecutive integers be $(x - 3), (x - 2), (x - 1), x, (x + 1), (x + 2), (x + 3)$

The reason we choose integers in this manner is so that the sum of the integers is a single term and does not have a constant added to it.

$$\text{Sum} = x - 3 + x - 2 + x - 1 + x + x + 1 + x + 2 + x + 3 = 7x$$

$$\frac{7x}{7} = 14$$

$$x = 14$$

Thus, the 7 consecutive integers are 11, 12, 13, 14, 15, 16 and 17.

Quick Tip: A way to answer this question is to recall that the average of an odd number of consecutive integers is the middle number in the ordered set.

6. The sum of three consecutive integers is zero. How many such sequences of integers exist?

Answer:

The possible sequences are $\{-1, 0, 1\}, \{-2, 0, 2\}, \{-3, 0, 3\}$ and so on.

Therefore, there are infinitely many such sequences.

7. Of the seven numbers $\sqrt{72}$, $\frac{3}{4}$, 6^2 , $\sqrt{0.25}$, $0.362596\dots$, $\frac{1}{5.2}$ and 0, which ones are irrational?

Answer:

$$\sqrt{72} = \sqrt{2 \times 2 \times 2 \times 2 \times 3 \times 3} = 6\sqrt{2}$$

$$0.362596\dots$$

$\sqrt{72}$ is an irrational number because it is the product of irrational and rational number

$0.362596\dots$ is irrational number because it cannot be written as a fraction (because no repeating pattern of digits exists, it was necessary to end its representation by an ellipsis).

8. Simplify $\sqrt{3^2 + 1^2}$

Answer:

$$\sqrt{3^2 + 1^2} = \sqrt{9 + 1} = \sqrt{10} = \sqrt{10}$$

9. Is $\frac{ab}{c}$, where $c < 0 < a < b$, negative or positive?

Answer:

The given inequality states that c is negative and a, b are positive. So, the product ab must be positive, and the denominator is negative.

When dividing a positive number by a negative number the result is a negative number.

Therefore, the expression must be negative.

10. If a, b, c are all > 0 , what is the sign of $(-a)^3 b^6 c^7$?

Answer:

$(-a)^3$ is negative and $b^6 c^7$ is positive

The product of a negative and positive number is negative.

Therefore, the sign of $(-a)^3 b^6 c^7$ is negative.

11. If the sum of 5 consecutive integers is 110, then find the smallest integer.

Answer:

Let the 5 consecutive integers be $(x - 2), (x - 1), x, (x + 1), (x + 2)$

$$\text{Sum} = x - 2 + x - 1 + x + x + 1 + x + 2 = 110$$

$$5x = 110$$

$$x = 22$$

The smallest of the 5 consecutive integers is $(x - 2) = 22 - 2 = 20$.

12. If $|x + 3| = 5$, what is x ?

Answer:

$$|x + 3| = 5$$

$$x + 3 = 5 \text{ or } x + 3 = -5$$

$$x = 2 \text{ or } x = -8.$$

Note that there must be two answers in this type of problem.

13. What is the minimum value that x can be when $|3x - 5| = 11$?

Answer:

$$|3x - 5| = 11 \quad [\quad 3x - 5 = 11 \quad \text{or} \quad 3x - 5 = -11$$

$$[\quad 3x = 16 \quad \text{or} \quad 3x = -6$$

$$[\quad x = \frac{16}{3} \quad \text{or} \quad x = \frac{-6}{3} = -2$$

The minimum value that x can take is -2 .

14. What is the value of $|x - 3| \times |x - 2| \times |x - 1|$ when $x = 1$?

Answer:

$$|x - 3| \times |x - 2| \times |x - 1| = |1 - 3| \times |1 - 2| \times |1 - 1| = |-2| \times |-1| \times |0| = 2 \times 1 \times 0 = \text{Zero.}$$

A short approach to this is to notice that since the expression is a product, and the last factor, $|x - 1|$, is 0 when $x = 1$, then the final product will be zero no matter what values come before the last factor.